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Cobb–Douglas Economy Model Optimum Production Control with Fuzzy Logic Principles

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Abstract Production function evaluation plays a fundamental role in economic relations for optimal solutions. The Cobb–Douglas (C–D) production function is widely used to mathematically describe the nonlinear relationship between labor and capital. Various solution alternatives exist in the literature, including traditional regression analysis, artificial neural networks (ANNs), genetic algorithms (GAs), and mathematical optimization procedures. Each of these approaches relies on dense bivalent (two-valued) logical mathematical expressions and architectural structures that provide approximate precise solutions. This paper presents three algorithms for optimal C–D function evaluation, including interval, bivalent, and fuzzy logic parameter representations, and the comparative calculation of the final solutions between them. The proposed methodology encompasses two-valued logic labor estimation with membership degree 1 and multiple fuzzy logic estimations through fuzzy sets with a range of risk levels. In this study, triangular fuzzy membership degree functions are adopted as fuzzy sets. This same approach demonstrates how fuzzy logic approximation of fundamental economic variables produce not only the most probable solution but also a set with the desired membership degree. The appeal of fuzzy logic mathematics lies in the ability of numerical data as well as the ability to combine all kinds of rational-based expert language experience with intuitive inference. Furthermore, in fuzzy logic, depending on the membership degrees, the solution set is represented by a probability cumulative distribution function (CDF) that provides reliability and risk values at any given probability level.

Keywords Cobb–Douglas · Economy · Fuzzy · Optimum · Production · Set

1 Introduction

In any economy, production is the fundamental process that drives the profitability and sustainability of the system. However, uncertainties exist in the forms of imprecision, missing data, and errors, and therefore, single numerical values, while precise, can yield completely unreliable results. To account for the uncertainty properties of economic variables, some researchers have favored probabilistic, statistical, and stochastic procedural methodologies, which cannot be reliably used in many applications if enough data are available. Hu [1] proposed the use of an interval mathematical computation approach with the optimum solution of the relevant economic functions. The Cobb–Douglas [2] (C–D) production function has been widely used in research on economics [3–5] and technological progress with positive assumptions including diminishing marginal products of labor and capital, constant returns to scale, the absence of unobservable inputs, and perfect competition [6, 7]. Restrictive assumptions that lack any flexibility (fuzziness) result in the label and capital values being constrained between zero and one [1]. These assumptions have not been validated by empirical observations and measurements. Harrison [8] noted that the labor share is transitory over time. This point was observed in various countries, such as Germany, France, Italy, and Spain [9] between 1970 and 1996; [10] reported a decline in labor shares in the United States, Germany, France, Italy, the Netherlands, and

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Belgium due to lower labor intensity. Hossain et al. [11] proposed an application of the nonlinear C–D production function, which provides several different inputs to a problem and ultimately produces a single output. These inputs can be two or more depending on the factors. Wang and Fu [12] discussed a special type of production function known as the C–D production function, based on empirical studies [2] and applied to optimal production economics. A new algorithm was developed to generate geographic C–D models by combining the concept of allometric scaling, used to estimate the parameters of the C–D function, with its application in transportation problems in China [13]. Proper management of a country's resources is a crucial issue for its economic development. Similarly, optimization of water management for three industries based on water demand forecasting, which is subject to many uncertainties, was addressed using the C–D production function. Muhammad et al. [14] discussed the C–D efficient production function in industrial optimization and estimated three productive parameters by modeling a nonlinear optimization using the Lagrange multiplier based on the least squares approach. The cost function is calculated by the linear function. Dinc et al. [15] proposed genetic algorithms (GAs) as an innovative application of the C–D production function for optimization. They examined the potential relationship between production output, labor input, and capital by integrating traditional optimization methods such as partial derivatives with evolutionary computational techniques to address complex economic constraints.

In the open literature, there are few advanced methods for solving C–D generation functions. Most of these rely on machine learning procedures using readily available software programs without considering the underlying logical aspects. The vast majority of those available rely on bivalent logic expressed through mathematical equations.

An effective solution of the C–D production function calculation is based on different production factors, supported using the ordinary least square method, the Lagrange multiplier procedure, and the constrained optimization problem for production [14]. In another study, genetic algorithm (GA) methodology was used in C–D production function optimization to examine the relationship between input labor and capital, and thus complex economic constraints were addressed by integrating traditional optimization methods through partial derivative evolutionary computation techniques [15]. These recent and older C–D production function solutions are based on bivalent logic and therefore they are all dependent on mathematical equations.

There are few studies in the literature regarding C–D generation function solutions for fuzzy logic applications. This study focuses on solving the C-D generation function using fuzzy AHP indexing, in accordance with the linear multiple regression analysis method, based on the C-D hybrid modeling approach used in determining the value of residential land plots in Konya/Türkiye using the C–D hybrid modeling approach [16]. This study's approach to handling the C–D production function using explicit fuzzy logic and representing input and capital variables in terms of membership functions offers a unique opportunity for the application of both linguistic experience and numerical dataset compositions.

Lukač et al. [17] attempted to relate optimal taxation and profit maximization problems using a bi-level model that treats the action of the leader government and the follower as a profit maximizer, using the C–D profit model. On the other hand, Lukač [18] formulized a government revenue maximizing textual revenue using a bi-level programming program, where the firm's output is defined by the C–D production function. The properties of the text revenue function are also examined by two numerical examples. Avvakumov et al. [19] presented strict concavity and constant elasticity of substitution (CES) formulation conditions for C–D. They have also considered the dependence of maximum profit on elasticity coefficients. Their paper also represented a topic relevant to the study of enterprises innovation activities. On the other hand, a gray C–D procedure was presented that helps estimate grand domestic production (GDP) based on the exchange rate of capital, and labor inputs for economic growth, where capital and labor are the key elements [20]. They also found the production elasticities of capital and labor inputs. Finally, they argued through empirical studies that the percentage increase in GDP is greater than the increase in capital stock and labor. Songur and Elmas [21] reviewed the existing literature on four different procedures, including C–D production function is overwhelmingly preferred in the literature for estimating output elasticity.

The C–D production function assumes a constant share in output, which can be too restrictive for a converging country [22]. They tested the reliability of the C–D production function for the Czech Republic economy and found no significant difference between the C–D production function and a more general supply-side production function calculation for the Czech economy for the period 1995–2005. The most general form of the C–D function remains influential in theoretical and empirical analyses of growth and productivity. Felipe and Adams [23] experimentally confirmed that the C–D function can reproduce the income accounting identity, where value added equals the sum of payroll and total profits. Ishikawaet al. [24] proposed a C–D procedure for the conceptualization of a symmetric plane in geography. They measured capital shares (capital elasticity) and labor shares (labor elasticity) and compared these with the results of multiple regression analysis used in economics.

This main purpose of this paper is to make a fundamental departure from the literature to demonstrate the applicability of fuzzy logic inference to the C–D production function by using fuzzy set membership degree representations of input and output variables. This approach is compared with single and interval numerical methodologies based on bivalent (two-value) logic, which are largely present in the relevant literature. It is demonstrated that the fuzzy methodology encompasses all applications of bivalent logic approaches. Furthermore, the rationale for this research is to address the uncertainty in economic parameters, which has not been explained in previous literature, through a fuzzy inference system.

2 Crisp and Fuzzy Logic

The primary objective of this paper is to provide evaluations of fuzzy logic economic equations that involve estimating the exact number, N, and the interval number [N_L, N_U] as specific forms. In Fig. 1a, the crisp logic (bivalent logic) solution produces a single numerical value with a degree of membership of one, i.e., absolute certainty; interval numerical value calculations produce an integrated interval result where the degree of membership of all-encompassing values equal to one (Fig. 1b), and therefore, there is no way to define the optimal solution. However, the fuzzy logic method adds a fuzzy set to each variable, and then mathematically manipulates the corresponding functions, resulting in a set of values for which one has the maximum membership function (Fig. 1c). Fuzzy mathematical calculations provide interval estimation solutions as well as solutions desired by economist. This paper proposes that the most efficient and logically reasonable approach is the fuzzy set calculation methodology by means of fuzzy inference system (FIS).

In economics, the general equilibrium expression includes a production function that implicitly relates output (quantity, Q) to input factors (labor, L, and capital, K; these are controllable variables).

$$Q = f(L, K). \tag{1}$$

Here, L is the amount of labor required to produce Q and can be expressed as labor hours expended in a month. On the other hand, another independent input factor, K, is the second input variable with units in hours to produce Q (e.g., machine hours).

Logically, both labor and capital are simultaneous prerequisites for production, and therefore output is directly proportional to Q. Consequently, their products play the most important role in defining Q. However, they are not linearly related to the quantity produced. Based on these logical expressions, various researchers have proposed explicit forms of the production function, the most famous of which is the C–D type, expressed in the following equation:

$$Q = \gamma L^{\alpha} K^{\beta}, \tag{2}$$

where Q = production rate; L = labor input; K = capital/equipment input; γ = technology and scale; α and β are the output elasticities of labor and capital, respectively. Hu [1] has stated that $\alpha + \beta = 1$ in many real-world situations. Each of these parameters depends on a firm's economic activity and must be determined empirically. A graphical representation of these two inputs and one output (2I1O) is shown in Fig. 2.

This figure shows a concave surface of the production function based on the arguments L and K. If appropriate for some calculations, Eq. (2) can also be written in logarithm form as follows:

$$\log Q = \log \gamma + \alpha \log L + \beta \log K \tag{3}$$

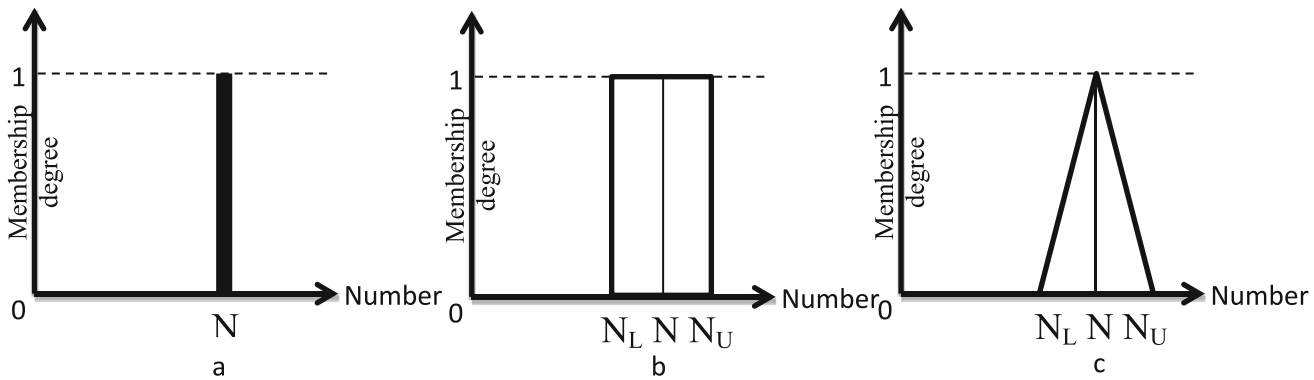


Fig. 1 Crisp, interval, and fuzzy numbers

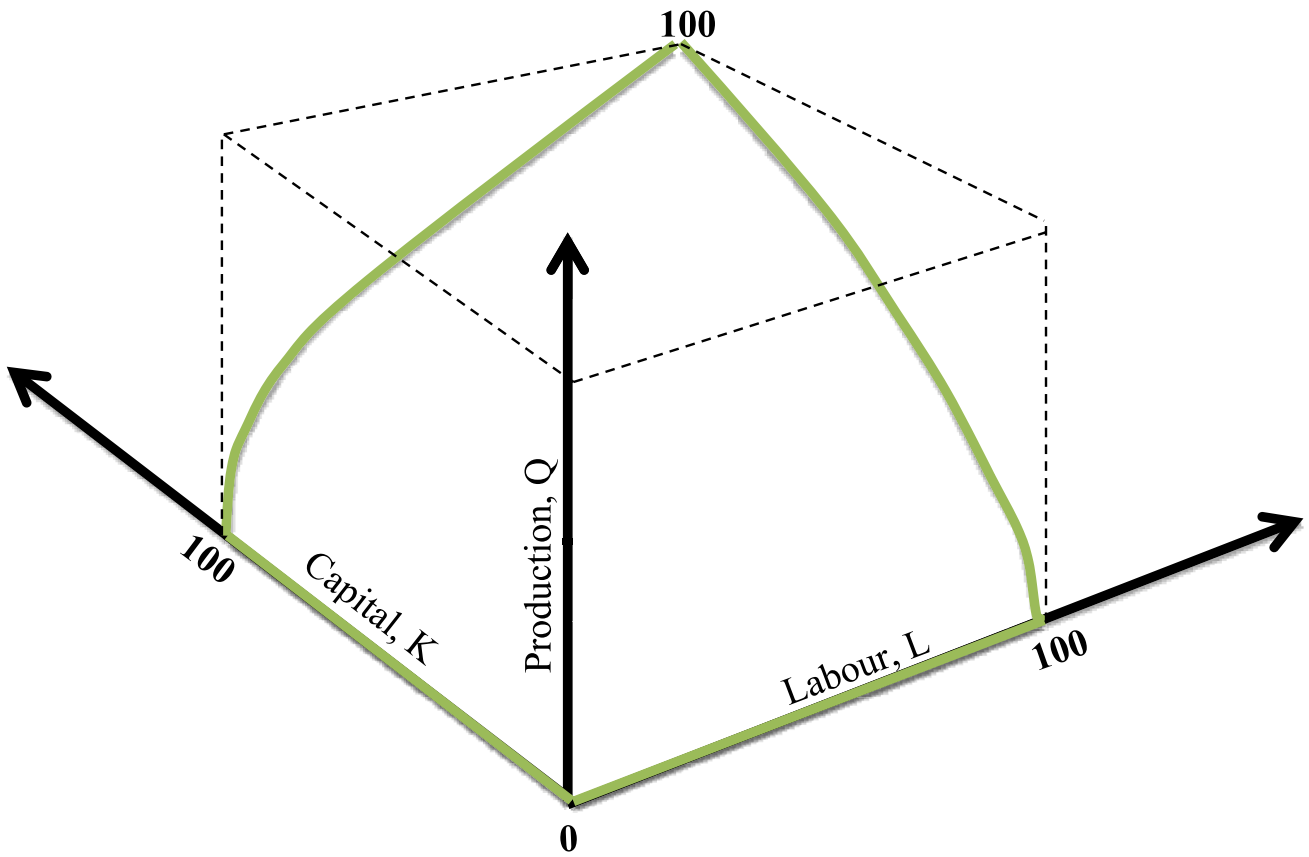


Fig. 2 Triple relationships in economic production function

In this case, the logarithmic relations provide a linear proportional expression in triple logarithmic space that is no different from Eq. (2) in terms of economic calculations.

3 Economic Fundamentals and Crisp Optimization

Besides the production function, another fundamental formulation in any economic model is the cost function, C, in terms of labor and capital, which can be written linearly as follows:

$$\mathrm{C} = \mathrm{aL} + \mathrm{bK}, \tag{4}$$

where a and b are the unit prices of labor and capital, respectively. Each firm seeks cost–benefit optimization (maximization) based on its production rate. In any optimization problem, maximizing the production function (Eq. 2) based on costs (Eq. 4) can be mathematically accomplished using the Lagrange multiplier, μ, leading to the objective function, O, as follows:

$$O = \gamma L^{\alpha} K^{\beta} - \mu(aL + bK). \tag{5}$$

To find the optimal solution, this objective function must be differentiated with respect to a and b, and then, by setting each result to zero, optimum control for the equilibrium market can be obtained through algebraic manipulations as follows:

$$\mathrm{L} = \left(\frac{\mathrm{Q}}{\mathrm{A}}\right)\left(\frac{\alpha}{\beta}\frac{\mathrm{b}}{\mathrm{a}}\right)^{\beta} \tag{6}$$

with

$$\mathrm{K} = \left(\frac{\mathrm{Q}}{\mathrm{A}}\right)\left(\frac{\beta}{\alpha}\frac{\mathrm{a}}{\mathrm{b}}\right)^{\beta}. \tag{7}$$

Optimum control for the seller's market can be obtained as follows:

$$\mathrm{L} = \frac{\alpha C}{\mathrm{a}} \tag{8}$$

and

$$K = \frac{\beta C}{b}, \tag{9}$$

respectively. In guiding these expressions, the condition $\alpha + \beta = 1$ is considered in the range of capital optima for the equilibrium and seller's markets.

4 Interval Assessments

In most economic studies, although the various parameters are not known with certainty, the parameter values are generally assumed to be precise, and therefore, substituting them into the existing equations (Eqs. 1–9) yields crisp (bivalent logic) results in terms of a single value. However, no economic parameter can be known with absolute certainty, and an economist can only approximate the most precise value possible. In economic research, experts often rely on approximate reasoning and therefore consider not a precise value but a range of parameter values within an interval with lower and upper bounds, including the decision value. In real life, parameters are known only approximately. An expert knows, at best, all possible values of the parameter within the interval. However, all values within this interval have the same degree of belonging, that is, the degree of membership (belonging), which in crisp logic (two-valued) is equal to one. In general, any interval parameter, for example A, will have lower bounds, $\underline{A}$ and upper bounds, $\overline{A}$. In addition to these extreme values, the most frequently used proxy for this range parameter is the arithmetic mean $\left(\underline{A}+\overline{A}\right)/2$, which corresponds to the mid-range value. Furthermore, different values within the range result in different labor or capital values. Decision makers will rely on their own experience and expertise and intuition to subjectively select the most appropriate value. However, different decision-makers will arrive at different values, which can lead to disagreements and debates. To eliminate such subjectivity, fuzzy sets can be proposed for a more systematic decision-making process.

5 Fuzzy Set Assessment

Social transitions occur in complex worlds such as economics, management, engineering, medicine, and science. The verbal forms of uncertainty are ambiguity, vagueness, incompleteness, skepticism, and fuzziness. In many disciplines, decision-makers rely on approximate reasoning to determine outcomes due to the pervasive complexity and uncertainty that permeate social, economic, political, and even technical problems. Scientific algorithms and equations are quite useful in this context, but they rely on assumptions to distract from the uncertainty content. This is a requirement for digital computers because they rely on equations and numerical calculations without verbal or linguistic knowledge sources. Fuzzy set logic and inference tools exist in the literature that process linguistic uncertainty and then transfer it to computers to reach objective conclusions [25–28]. Fuzzy sets provide an objective basis for computers to address complex scientific, economic, social, environmental, and political problems. The number of fuzzy applications, including fuzzy consumer products and new patents is rapidly increasing.

Although economic variables such as labor, capital, and production are not known with certainty in terms of parameters such as cost, "C," the parameter "a" represents the cost of labor power, the parameter "b" represents the cost of acquiring a unit of capital, and α represents the cost of acquiring a unit of capital. The fuzzy set derivation of the labor variable in Eq. (8) is discussed below. The constants dependent on a firm's output must be determined empirically. In any dynamic environment, even a fixed value will carry numerical and, to some extent linguistic (verbal) uncertainties. Although Hu [2] addresses these uncertainties by considering intervals for each parameter, provided that no number dominates the other, in this paper, "C," "α," and "a" are referred to as fixed parameters. Each of these so-called fixed parameters is represented by fuzzy sets (see Fig. 3).

In this paper, each constant is represented by a triangular fuzzy number (m_c, m_α, m_a) according to the following formulations, where x is a dummy variable for each constant. After establishing the general structure of fuzzy logic membership functions in the form of triangular sets, replacing these sets with other types such as Trapezoidal, Gaussian, Generalized bell, Sigmoidal is an automatic process to reach alternative solutions.

$$m_C = \begin{cases} 0 & x \le C_l \\ x - C_l & C_l < x \le C_m \\ -x + C_u & C_m \le x < C_u \\ 0 & x \ge C_u \end{cases}, \tag{10}$$

$$m_\alpha = \begin{cases} 0 & x \le \alpha_l \\ x - \alpha_l & \alpha_l < x \le \alpha_m \\ -x + \alpha_u & \alpha_m \le x < \alpha_u \\ 0 & x \ge \alpha_u \end{cases}, \tag{11}$$

and

$$m_a = \begin{cases} 0 & x \le a_l \\ x - a_l & a_l < x \le a_m \\ -x + a_u & a_m \le x < a_u \\ 0 & x \ge a_u \end{cases}. \tag{12}$$

It should be noticed that each constant has its own lower bound (C_l, α_1, and a_l), medium (C_m, α_m, and a_m), and upper bound (C_u, α_u , and a_u). Each of these numbers depends on the firm's operating production and management systems and expert opinions. After summing these numbers, all that remains is to perform the fuzzy set arithmetic operations, which lead to the detailed algebraic calculations presented in the Appendix. Every fuzzy logic rule base proposition has the structure "IF…(Premise)..THEN…(Conclusion)." The "premise" part contains the relationships of the input

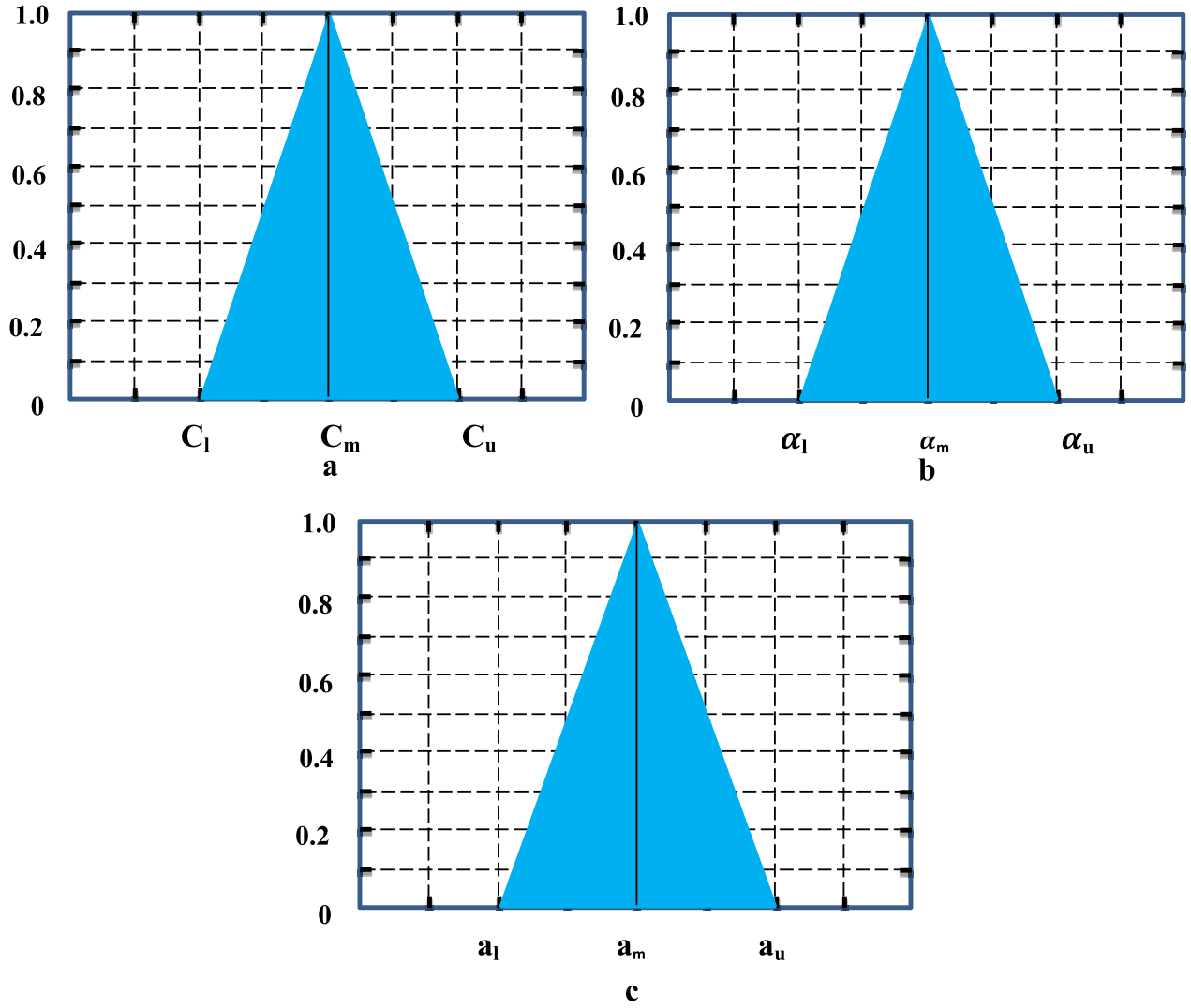


Fig. 3 Fuzzy set of **a** "C" constant, **b** "α" constant, **c** "a" constant

variables through fuzzy sets, which leads to the output fuzzy set, which is the corresponding "conclusion" part. The input fuzzy sets are related to each other by the conjunction "AND," and each fuzzy proposition is related to the others by the conjunction "OR." The final fuzzy set form of the labor variable is obtained as follows:

$$\frac{m_C x m_\alpha}{m_a} = \begin{cases} 0 & \leq \dfrac{C_l \alpha_l}{a_u} \\ \dfrac{-(C_l+\alpha_l+x)+\sqrt{(C_l+\alpha_l+x)^2-4(C_l\alpha_l-a_u x)}}{2} & \dfrac{C_l\alpha_l}{a_u} \leq x < \dfrac{C_m\alpha_m}{a_m} \\ \dfrac{(C_u+\alpha_u+x)-\sqrt{(C_u+\alpha_u+x)^2-4(C_u\alpha_u-a_l x)}}{2} & \dfrac{C_m\alpha_m}{a_m} < x \leq \dfrac{C_u\alpha_u}{a_l} \\ 0 & x \geq \dfrac{C_u\alpha_u}{a_l} \end{cases} \tag{13}$$

The first striking point in the final labor fuzzy set is the lack of symmetry, even when each of the basic gradients is combined with the triangular and symmetric fuzzy sets (see Fig. 4). In this way, the point, interval, and fuzzy estimation alternatives are collectively suppressed. The highest

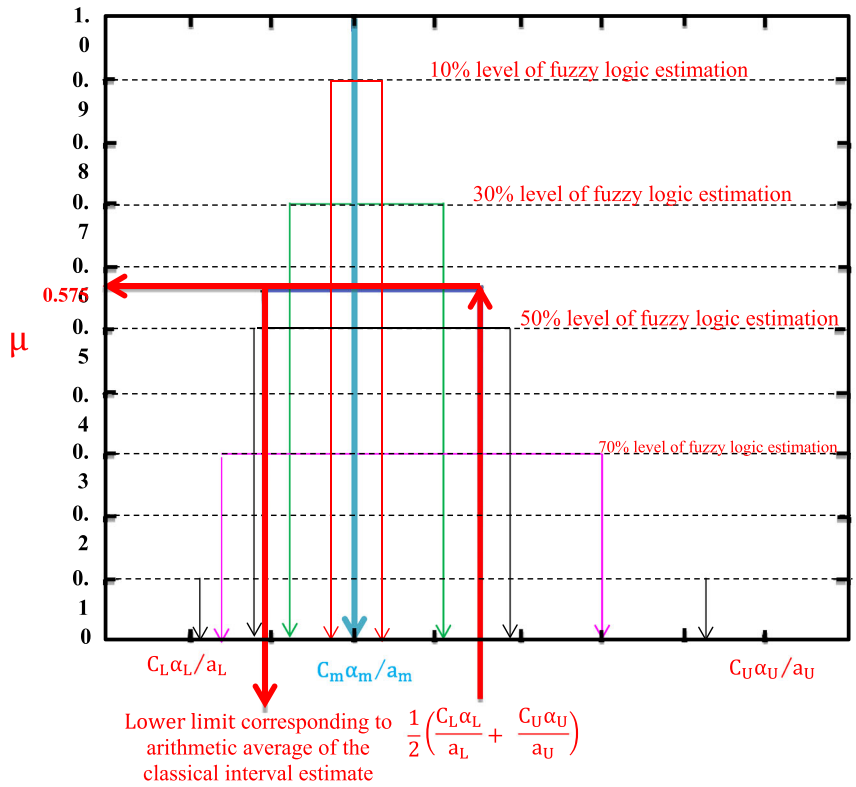


Fig. 4 Point, interval, and fuzzy estimation alternatives

Table 1 Interval estimates at different membership degree levels

Type of methodology	Membership level	Fuzzy set interval estimates	
		Lower limit	Upper limit
Crisp	1.0	$C_m\alpha_m/a_m$	
F	0.9	$2.8C_L\alpha_L/a_L$	$3.3C_L\alpha_L/a_L$
U	0.7	$2.3C_L\alpha_L/a_L$	$4.1C_L\alpha_L/a_L$
Z	0.5	$1.9C_L\alpha_L/a_L$	$4.9C_L\alpha_L/a_L$
Z	0.3	$1.3C_L\alpha_L/a_L$	$5.0C_L\alpha_L/a_L$
Y	0.1	$1.1C_L\alpha_L/a_L$	$7.3C_L\alpha_L/a_L$
Interval	0.0	$C_L\alpha_L/a_L$	$C_u\alpha_u/a_u$

membership degree in point estimation is set to 1, resulting in the value of $c_m\ \alpha_m/a_m$. Classical interval estimate has a variation range between $c_l\ \alpha_l/a_l$ and $c_u\ \alpha_u/a_u$ with membership degrees equal to zero. In this classical interval approach, there is no possibility of selecting any labor value between these extreme values. In practice, the best that can be done is to accept the midpoint of $0.5(c_L\ \alpha_L/a_L + c_U\ \alpha_U/a_U)$ as the final decision value. The arithmetic value of this membership degree or this midpoint is approximately 0.575, but in classical interval estimation, the researcher assumes the membership value is 1 due to the principle of crisp logic (bivalent logic).

Therefore, the reliability of classical interval and fuzzy logic estimation is approximately 0.425. Furthermore, considering fuzzy set inference, the lower bound of the same arithmetic mean appears to be a value close to the classical interval limit $c_L\ \alpha_L/a_L$. This lower bound can be calculated using the algebra shown in the Appendix and it can be seen that there are an infinite number of fuzzy estimates corresponding to any desired membership degree between the maximum (1) and minimum (0) membership degrees.

The fuzzy set approach also provides crisp point estimate with degree of membership equal to 1, but several different interval estimates exist around this sharp point estimate at each membership level. As an example, Table 1 contains a series of fuzzy estimates at various membership degrees (0.90, 0.70, 0.50, 0.30, 0.10, and 0).

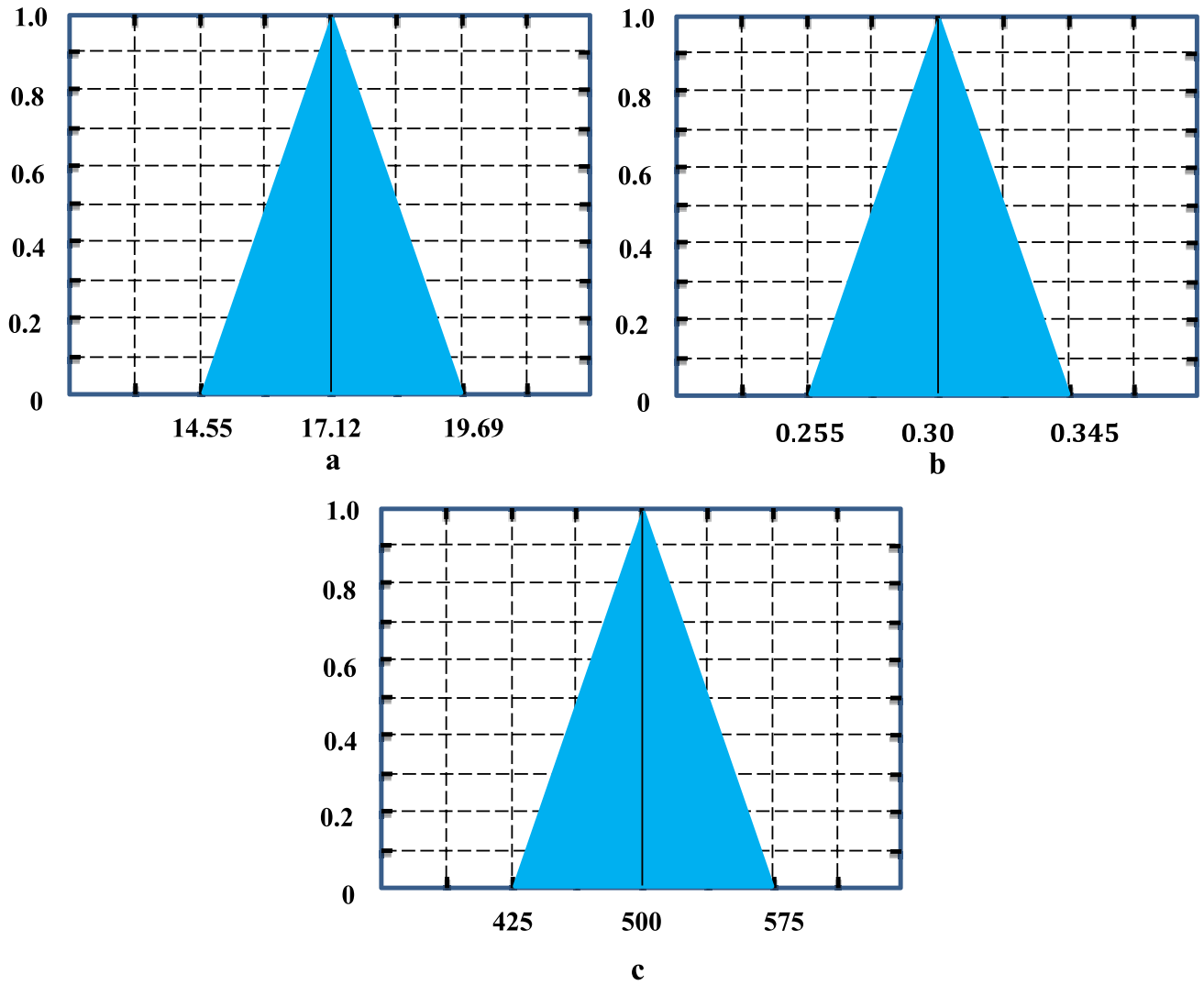


Fig. 5 Triangular fuzzy sets for each variable

Another noteworthy point is that, around the fuzzy point estimate at each membership degree level across the interval estimates, the lower bound is relatively closer to the fuzzy point estimate than the upper bound. This means that under uncertainty, the upper bounds of the labor quantity have more uncertainty than the lower bounds.

6 Numerical Example

A case study is presented to illustrate how the proposed framework works. The case results are presented based on the fuzzy values of the parameters ($\alpha, \beta,$ a and b) of the C–D function.

In most existing production studies [29], $\alpha = 0.3$, *and*$\beta =$ 0.7 are frequently reported. Since the specific values of γ, α and β are considered state parameters and will not affect the overall proposed framework, for practical purposes, $\gamma =$ 1, = 0.3, = 0.7 is assumed here. Therefore, Eq. (2) can be written as follows:

$$Q = L^{0.3}K^{0.7}. \tag{14}$$

On the other hand, the constants of the cost function in Eq. (4) are a and b, which are the wage rate and the equipment rental rate, respectively. In practice, the labor cost is set at a = 500 Turkish Lira (TL) per unit labor and the capital cost is set at b = 1250 TL per unit of equipment.

Considering the previous assumptions, to provide a better understanding of the fuzzy set approach for labor, the following numerical example is adapted from [30], who wrote an article on the beneficial effects of the carbon tax on the Turkish energy sector. He calculated the crisp values of the capital and labor amounts as approximately $K = 165 \times 10^6$ TL and $L = 71 \times 10^6$ TL in TL. Substituting these values into Eq. (4), we can obtain $C = 500 \times 165 \times 10^6 + 1250 \times 71 \times 10^6 = 17.12$

Table 2 Numerical solution for each methodology

Type of methodology	Membership level	Fuzzy set interval estimates	
		Lower limit ($\times 10^{-3}$)	Upper limit ($\times 10^{-3}$)
Crisp	1.0	9.63	
F	0.9	9.54	9.81
U	0.7	9.27	10.05
Z	0.5	9.09	10.44
Z	0.3	8.99	10.97
Y	0.1	8.82	11.43
Interval	0.0	8.73	11.81

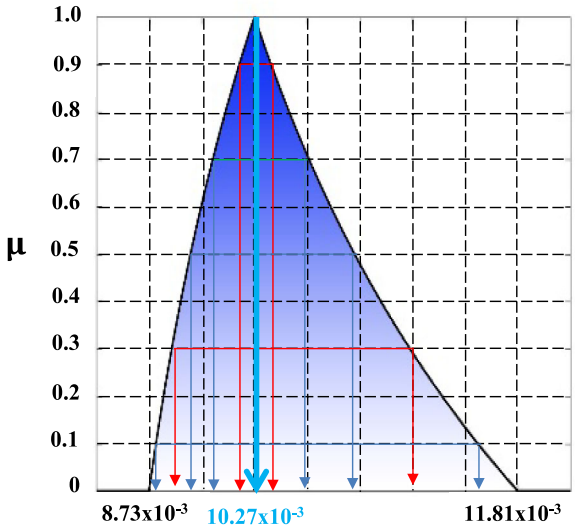


Fig. 6 Crisp, interval, and fuzzy estimation numerical values

$\times 10^{10}$ TL. The fuzzy sets for each C,α , and a parameter can be shown in Fig. 5, where lower and upper bounds are set to $\pm 15\%$ deviation from the crisp logic values.

The graph obtained according to the algebraic calculation of fuzzy numbers described previously is shown in Fig. 6, and the numerical representations of the crisp, interval, and fuzzy estimated values are shown in Table 2.

This table compares three methodologies—crisp, interval, and fuzzy logic solutions for the example given in the text. The crisp (binary) logic solution is 9.63×10^{-3}, while the interval solution ranges from 8.73×10^{-3} to 11.81×10^{-3}. As explained in the introduction, while most similar studies rely on binary logic with a single solution value, fuzzy logic offers a range of solutions with any desired degree of membership (Table 2), which can be represented as the probability cumulative distribution function (CDF) given in Fig. 7.

In this way, the lower and upper bounds of the fuzzy logic cumulative distribution function (CDF) curves are given, and these are useful in providing accurate results in terms of risk and reliability for an overshoot probability of 0.70, for example, as shown in the figure. On the vertical axis on the right, the complementary reliability and risk value ranges are shown at this probability level, and the result is between 9.28×10^{-3} and 10.86×10^{-3}. Depending on the risk and reliability percentages, the lower and upper bounds of the fuzzy logic solution can be determined.

7 Dynamic Optimum Control

In real life, decisions regarding economic issues are assumed to be made in a dynamic environment dependent on local and international conditions. This dynamism, in its simplest form, occurs between labor and capital. Given available capital resources, a production unit may choose to replace some of its labor input with some of its capital. This means that fundamental economic parameters change over time. The possibility of substitution represents the range of parameter values. The optimal decision parameter value at one time may not be valid at another, and therefore, the update is made according to the new conditions. When the optimality of a parameter does not apply to all economic parameters involved in the decision-making process, decision-maker must select parameter values.

For example, consider the market situation of a seller, where a simple parameter update can be performed as follows. In general, Fig. 4 shows all possible options depending on the conditions. It is the decision-maker's responsibility to verify whether the current parameters are still the best decision. To this end, the following steps are necessary to achieve the new optimality.

(a) Assume that the previous labor is represented by L_P level, and the new situation is presented by the L_N level. If these two levels are not equal, a new decision is needed,

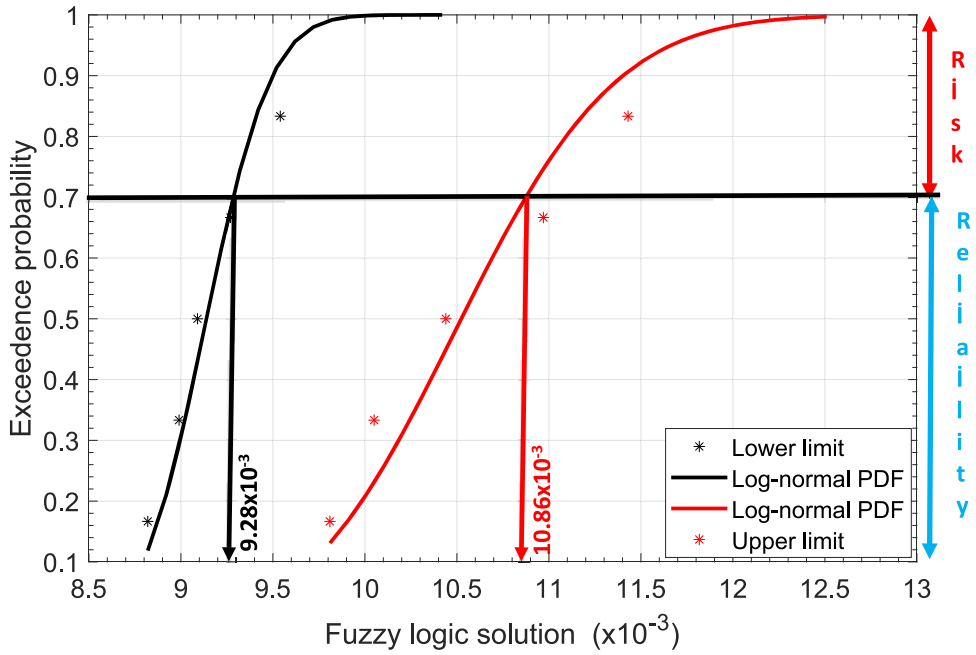


Fig. 7 Fuzzy logic solution CDF

(b) If the L_P and L_N values are close within relative error margin ± 5%, the final decision is given as the most optimum result.

8 Conclusions

In any firm's economic evaluation, the two most fundamental variables are labor and capital, which are crucial for the firm's production and the role of these variables in customer spending. The firm's (salesperson's) production function is a nonlinear function of two variables (capital and labor) and is often expressed using the Cobb–Douglas (C–D) expression, while costumer spending is their linear sum. Optimizing a firm's output by considering costumer spending, or vice versa, requires decision-makers to express labor and capital using in three constants that must be estimated empirically and intuitively. In practical applications, these parameters are treated as fixed numbers. However, there are also studies that treat them as interval numbers in crisp (bivalent) logic sets without prioritizing the representation of the relevant economics. This study uses a fuzzy logic set approach, where reasonable intervals are established between representative numbers based on the degrees of membership for each parameter. Each parameter is processed through a triangular fuzzy set function, and the optimum labor force is expressed after the fuzzy set mathematical operation that includes fuzzy rule base. In this process, an integrated fuzzy set is obtained where each labor force value has a degree of membership. The integrated fuzzy set includes a crisp (single) labor estimate with of membership degree 1; crisp interval estimates, and finally, multiple fuzzy estimates of the interval between them according to the degree of risk level. Similar studies can be conducted in the future with different membership functions other than triangular sets.

Appendix

$$y = a_1^{\beta} - C_l \rightarrow a_1^{\beta} = y + C_l$$

$$y = -a_2^{\beta} + C_u \rightarrow a_2^{\beta} = C_u - y$$

$$C_y = [y + C_l, C_u - y]$$

Same substation for α and a

$$\alpha_y = [y + \alpha_l, \alpha_u - y]$$

$$a_y = [y + a_l, a_u - y]$$

$C_y x \alpha_y$ fuzzy multiplication

$$C_y x \alpha_y = [y + C_l, C_u - y] \cdot [y + \alpha_l, \alpha_u - y]$$

$$= \left[y^2 - (C_l + \alpha_l)y + C_l\alpha_l, y^2 - (C_u + \alpha_u)y + C_u\alpha_u\right]$$

$$y^2 - (C_l + \alpha_l)y + C_l\alpha_l - x = 0$$

$$y = \frac{(C_l + \alpha_l) - \sqrt{(C_l + \alpha_l)^2 - 4(C_l\alpha_l - x)}}{2}$$

$m_C x m_\alpha$ multiplication obtained as follows:

$$m_C x m_\alpha = \begin{cases} 0 & x \le C_l\alpha_l \\ \dfrac{-(C_l+\alpha_l)+\sqrt{(C_l+\alpha_l)^2-4(C_l\alpha_l-x)}}{2} & C_l\alpha_l \le x < C_m\alpha_m \\ \dfrac{(C_u+\alpha_u)-\sqrt{(C_u+\alpha_u)^2-4(C_u\alpha_u-x)}}{2} & C_m\alpha_m < x \le C_u\alpha_u \\ 0 & x \ge C_u\alpha_u \end{cases}$$

Fuzzy membership function multiplication

$$z = \frac{-(C_l+\alpha_l)+\sqrt{(C_l+\alpha_l)^2-4\left(C_l\alpha_l-a_1^\beta\right)}}{2} \to a_1^\beta$$
$$= C_l\alpha_l + z(z+C_l+\alpha_l)$$

$$z = \frac{(C_u+\alpha_u)-\sqrt{(C_u+\alpha_u)^2-4\left(C_u\alpha_u-a_2^\beta\right)}}{2} \to a_2^\beta$$
$$= C_u\alpha_u - z(C_u+\alpha_u-z)$$

$$(m_C x m_\alpha)_z = [(C_l\alpha_l + z(z+C_l+\alpha_l)), (C_u\alpha_u - z(C_u+\alpha_u-z))]$$

$$a_z = [z+a_l, a_u - z]$$

$$\frac{(m_C x m_\alpha)_z}{(m_a)_z} = \left[\frac{(C_l\alpha_l + z(z+C_l+\alpha_l))}{a_u - z}, \frac{(C_u\alpha_u - z(C_u+\alpha_u-z))}{z+a_l}\right]$$

For 1st interval:

$$\frac{(C_l\alpha_l + z(z+C_l+\alpha_l))}{a_u - z} - x = 0$$

Let us substitute $(C_l+\alpha_l) = A_l$;

$$\frac{(C_l\alpha_l + z(z+C_l+\alpha_l))}{a_u - z} - x = 0$$

$$z = \frac{-(C_l+\alpha_l+x)+\sqrt{(C_l+\alpha_l+x)^2-4(C_l\alpha_l-a_u x)}}{2}$$

For 2nd interval:

$$\frac{\left(C_u\alpha_u - \frac{(C_u+\alpha_u)^2-\left[(C_u+\alpha_u)-2z\right]^2}{4}\right)}{z+a_l} - x = 0$$

Let us substitute $(C_u+\alpha_u) = A_u$;

$$\frac{(C_u\alpha_u - z(C_u+\alpha_u-z))}{z+a_l} - x = 0$$

$$z = \frac{(C_u+\alpha_u+x)-\sqrt{(C_u+\alpha_u+x)^2-4(C_u\alpha_u-a_l x)}}{2}$$

$$\frac{m_C x m_\alpha}{m_a} = \begin{cases} 0 & \frac{C_l\alpha_l}{a_u} \\ \dfrac{-(C_l+\alpha_l+x)+\sqrt{(C_l+\alpha_l+x)^2-4(C_l\alpha_l-a_u x)}}{2} & \frac{C_l\alpha_l}{a_u} \le x < \frac{C_m\alpha_m}{a_m} \\ \dfrac{(C_u+\alpha_u+x)-\sqrt{(C_u+\alpha_u+x)^2-4(C_u\alpha_u-a_l x)}}{2} & \frac{C_m\alpha_m}{a_m} < x \le \frac{C_u\alpha_u}{a_l} \\ 0 & x \ge \frac{C_u\alpha_u}{a_l} \end{cases}$$

Fuzzy membership function division.

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Code Availability Whoever requests the code in this article, the author will provide it in Matlab form.

Declarations

Conflict of interest There is no conflict of interest.

Ethical Approval This new research paper has been conducted ethically, keeping in mind privacy, consent, and appropriate referencing of those related to the subject.

Consent to Participate This is a single author paper and all the work is made by the author.

Consent for Publication The informed consent document succinctly describes the research theoretical and subsequent application parts.

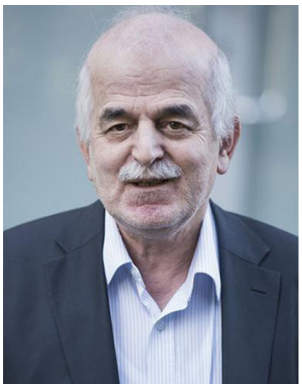
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